

Exercise 7.10

Question 1:

$$\int_0^1 \frac{x}{x^2+1} dx$$

Answer

$$\int_0^1 \frac{x}{x^2+1} dx$$

$$\text{Let } x^2 + 1 = t \Rightarrow 2x dx = dt$$

When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

$$\begin{aligned} \therefore \int_0^1 \frac{x}{x^2+1} dx &= \frac{1}{2} \int_1^2 \frac{dt}{t} \\ &= \frac{1}{2} [\log |t|]_1^2 \\ &= \frac{1}{2} [\log 2 - \log 1] \\ &= \frac{1}{2} \log 2 \end{aligned}$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^5 \phi d\phi$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^5 \phi d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^4 \phi \cos \phi d\phi$$

$$\text{Also, let } \sin \phi = t \Rightarrow \cos \phi d\phi = dt$$

When $\phi = 0$, $t = 0$ and when $\phi = \frac{\pi}{2}$, $t = 1$

$$\begin{aligned}
 \therefore I &= \int_0^1 \sqrt{t} (1-t^2)^2 dt \\
 &= \int_0^1 t^{\frac{1}{2}} (1+t^4-2t^2) dt \\
 &= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}} \right] dt \\
 &= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}} \right]_0^1 \\
 &= \frac{2}{3} + \frac{2}{11} - \frac{4}{7} \\
 &= \frac{154+42-132}{231} \\
 &= \frac{64}{231}
 \end{aligned}$$

Question 3:

$$\int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$$

Answer

$$\text{Let } I = \int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$$

Also, let $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{4}$

$$\begin{aligned}
 I &= \int_0^{\frac{\pi}{4}} \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \sec^2 \theta \, d\theta \\
 &= \int_0^{\frac{\pi}{4}} \sin^{-1} (\sin 2\theta) \sec^2 \theta \, d\theta \\
 &= \int_0^{\frac{\pi}{4}} 2\theta \cdot \sec^2 \theta \, d\theta \\
 &= 2 \int_0^{\frac{\pi}{4}} \theta \cdot \sec^2 \theta \, d\theta
 \end{aligned}$$

Taking θ as first function and $\sec^2 \theta$ as second function and integrating by parts, we obtain

$$\begin{aligned}
 I &= 2 \left[\theta \int \sec^2 \theta \, d\theta - \int \left\{ \left(\frac{d}{d\theta} \theta \right) \int \sec^2 \theta \, d\theta \right\} d\theta \right]_0^{\frac{\pi}{4}} \\
 &= 2 \left[\theta \tan \theta - \int \tan \theta \, d\theta \right]_0^{\frac{\pi}{4}} \\
 &= 2 \left[\theta \tan \theta + \log |\cos \theta| \right]_0^{\frac{\pi}{4}} \\
 &= 2 \left[\frac{\pi}{4} \tan \frac{\pi}{4} + \log \left| \cos \frac{\pi}{4} \right| - \log |\cos 0| \right] \\
 &= 2 \left[\frac{\pi}{4} + \log \left(\frac{1}{\sqrt{2}} \right) - \log 1 \right] \\
 &= 2 \left[\frac{\pi}{4} - \frac{1}{2} \log 2 \right] \\
 &= \frac{\pi}{2} - \log 2
 \end{aligned}$$

Question 4:

$$\int_0^2 x\sqrt{x+2} \, dx \quad (\text{Put } x+2=t^2)$$

Answer

$$\int_0^2 x\sqrt{x+2} \, dx$$

$$\text{Let } x+2 = t^2 \Rightarrow dx = 2t \, dt$$

When $x = 0$, $t = \sqrt{2}$ and when $x = 2$, $t = 2$

$$\begin{aligned}
 \therefore \int_0^2 x\sqrt{x+2}dx &= \int_{\sqrt{2}}^2 (t^2 - 2)\sqrt{t^2} 2t dt \\
 &= 2 \int_{\sqrt{2}}^2 (t^2 - 2)t^2 dt \\
 &= 2 \int_{\sqrt{2}}^2 (t^4 - 2t^2) dt \\
 &= 2 \left[\frac{t^5}{5} - \frac{2t^3}{3} \right]_{\sqrt{2}}^2 \\
 &= 2 \left[\frac{32}{5} - \frac{16}{3} - \frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3} \right] \\
 &= 2 \left[\frac{96 - 80 - 12\sqrt{2} + 20\sqrt{2}}{15} \right] \\
 &= 2 \left[\frac{16 + 8\sqrt{2}}{15} \right] \\
 &= \frac{16(2 + \sqrt{2})}{15} \\
 &= \frac{16\sqrt{2}(\sqrt{2} + 1)}{15}
 \end{aligned}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx$$

Let $\cos x = t \Rightarrow -\sin x \, dx = dt$ When $x = 0$, $t = 1$ and when $x = \frac{\pi}{2}$, $t = 0$

$$\begin{aligned}
 \Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx &= - \int_1^0 \frac{dt}{1+t^2} \\
 &= - \left[\tan^{-1} t \right]_1^0 \\
 &= - \left[\tan^{-1} 0 - \tan^{-1} 1 \right] \\
 &= - \left[-\frac{\pi}{4} \right] \\
 &= \frac{\pi}{4}
 \end{aligned}$$

Question 6:

$$\int_0^2 \frac{dx}{x+4-x^2}$$

Answer

$$\begin{aligned}
 \int_0^2 \frac{dx}{x+4-x^2} &= \int_0^2 \frac{dx}{-(x^2-x-4)} \\
 &= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)} \\
 &= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]} \\
 &= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}
 \end{aligned}$$

Let $x - \frac{1}{2} = t$ $\square$ $dx = dt$

When $x = 0$, $t = -\frac{1}{2}$ and when $x = 2$, $t = \frac{3}{2}$

$$\begin{aligned}\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} &= \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2} \\&= \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\frac{\sqrt{17}}{2} + t}{\frac{\sqrt{17}}{2} - t} \right]_{-\frac{1}{2}}^{\frac{3}{2}} \\&= \frac{1}{\sqrt{17}} \left[\log \frac{\frac{\sqrt{17}}{2} + \frac{3}{2}}{\frac{\sqrt{17}}{2} - \frac{3}{2}} - \log \frac{\frac{\sqrt{17}}{2} - \frac{1}{2}}{\frac{\sqrt{17}}{2} + \frac{1}{2}} \right] \\&= \frac{1}{\sqrt{17}} \left[\log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} - \log \frac{\sqrt{17} - 1}{\sqrt{17} + 1} \right] \\&= \frac{1}{\sqrt{17}} \log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} \times \frac{\sqrt{17} + 1}{\sqrt{17} - 1} \\&= \frac{1}{\sqrt{17}} \log \left[\frac{17 + 3 + 4\sqrt{17}}{17 + 3 - 4\sqrt{17}} \right] \\&= \frac{1}{\sqrt{17}} \log \left[\frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}} \right] \\&= \frac{1}{\sqrt{17}} \log \left(\frac{5 + \sqrt{17}}{5 - \sqrt{17}} \right) \\&= \frac{1}{\sqrt{17}} \log \left[\frac{(5 + \sqrt{17})(5 + \sqrt{17})}{25 - 17} \right] \\&= \frac{1}{\sqrt{17}} \log \left[\frac{25 + 17 + 10\sqrt{17}}{8} \right] \\&= \frac{1}{\sqrt{17}} \log \left(\frac{42 + 10\sqrt{17}}{8} \right) \\&= \frac{1}{\sqrt{17}} \log \left(\frac{21 + 5\sqrt{17}}{4} \right)\end{aligned}$$

Question 7:

$$\int_{-1}^1 \frac{dx}{x^2 + 2x + 5}$$

Answer

$$\int_{-1}^1 \frac{dx}{x^2 + 2x + 5} = \int_{-1}^1 \frac{dx}{(x^2 + 2x + 1) + 4} = \int_{-1}^1 \frac{dx}{(x+1)^2 + (2)^2}$$

Let $x + 1 = t \Rightarrow dx = dt$ When $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

$$\begin{aligned}\therefore \int_{-1}^1 \frac{dx}{(x+1)^2 + (2)^2} &= \int_0^2 \frac{dt}{t^2 + 2^2} \\ &= \left[\frac{1}{2} \tan^{-1} \frac{t}{2} \right]_0^2 \\ &= \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} 0 \\ &= \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{\pi}{8}\end{aligned}$$

Question 8:

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$$

Answer

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$$

Let $2x = t \Rightarrow 2dx = dt$ When $x = 1$, $t = 2$ and when $x = 2$, $t = 4$

$$\begin{aligned}\therefore \int_2^4 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx &= \frac{1}{2} \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2} \right) e^t dt \\ &= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt\end{aligned}$$

$$\text{Let } \frac{1}{t} = f(t)$$

$$\text{Then, } f'(t) = -\frac{1}{t^2}$$

$$\begin{aligned}\Rightarrow \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt &= \int_2^4 e^t [f(t) + f'(t)] dt \\ &= [e^t f(t)]_2^4 \\ &= \left[e^t \cdot \frac{2}{t} \right]_2^4 \\ &= \left[\frac{e^t}{t} \right]_2^4 \\ &= \frac{e^4}{4} - \frac{e^2}{2} \\ &= \frac{e^2(e^2 - 2)}{4}\end{aligned}$$

Question 9:

The value of the integral $\int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$ is

- A.** 6
- B.** 0
- C.** 3
- D.** 4

Answer

$$\text{Let } I = \int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$$

$$\text{Also, let } x = \sin \theta \Rightarrow dx = \cos \theta d\theta$$

$$\text{When } x = \frac{1}{3}, \theta = \sin^{-1}\left(\frac{1}{3}\right) \text{ and when } x = 1, \theta = \frac{\pi}{2}$$

$$\begin{aligned} \Rightarrow I &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta - \sin^3 \theta)^{\frac{1}{3}}}{\sin^4 \theta} \cos \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}} (1 - \sin^2 \theta)^{\frac{1}{3}}}{\sin^4 \theta} \cos \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}} (\cos \theta)^{\frac{2}{3}}}{\sin^4 \theta} \cos \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}} (\cos \theta)^{\frac{2}{3}}}{\sin^2 \theta \sin^2 \theta} \cos \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\cos \theta)^{\frac{5}{3}}}{(\sin \theta)^{\frac{5}{3}}} \operatorname{cosec}^2 \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} (\cot \theta)^{\frac{5}{3}} \operatorname{cosec}^2 \theta d\theta \end{aligned}$$

$$\text{Let } \cot \theta = t \quad \square \quad -\operatorname{cosec}^2 \theta d\theta = dt$$

When $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, $t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}$, $t = 0$

$$\begin{aligned}\therefore I &= -\int_{2\sqrt{2}}^0 (t)^{\frac{5}{3}} dt \\&= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^0 \\&= -\frac{3}{8}\left[(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^0 \\&= -\frac{3}{8}\left[-(2\sqrt{2})^{\frac{8}{3}}\right] \\&= \frac{3}{8}\left[(\sqrt{8})^{\frac{8}{3}}\right] \\&= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right] \\&= \frac{3}{8}[16] \\&= 3 \times 2 \\&= 6\end{aligned}$$

Hence, the correct Answer is A.

Question 10:

If $f(x) = \int_0^x t \sin t \, dt$, then $f'(x)$ is

- A.** $\cos x + x \sin x$
- B.** $x \sin x$
- C.** $x \cos x$
- D.** $\sin x + x \cos x$

Answer

$$f(x) = \int_0^x t \sin t \, dt$$

Integrating by parts, we obtain

$$\begin{aligned}f(x) &= t \int_0^x \sin t \, dt - \int_0^x \left\{ \left(\frac{d}{dt} t \right) \int_0^t \sin t \, dt \right\} dt \\&= \left[t(-\cos t) \right]_0^x - \int_0^x (-\cos t) dt \\&= \left[-t \cos t + \sin t \right]_0^x \\&= -x \cos x + \sin x\end{aligned}$$

$$\begin{aligned}\Rightarrow f'(x) &= - \left[\{x(-\sin x)\} + \cos x \right] + \cos x \\&= x \sin x - \cos x + \cos x \\&= x \sin x\end{aligned}$$

Hence, the correct Answer is B.