

Exercise 4.5**Question 1:**

Find adjoint of each of the matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We have,

$$A_{11} = 4, A_{12} = -3, A_{21} = -2, A_{22} = 1$$

$$\therefore \text{adj}A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Question 2:

Find adjoint of each of the matrices.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}.$$

We have,

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = - \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2 + 10) = -12$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$$

$$A_{21} = - \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1$$

$$A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1+4 = 5$$

$$A_{23} = - \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5-6 = -11$$

$$A_{32} = - \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$$

$$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3+2 = 5$$

$$\text{Hence, } \text{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}.$$

Question 3:

Verify $A (\text{adj } A) = (\text{adj } A) A = |A| I$.

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

Answer

$$A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

we have,

$$|A| = -12 - (-12) = -12 + 12 = 0$$

$$\therefore |A|I = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now,

$$A_{11} = -6, A_{12} = 4, A_{21} = -3, A_{22} = 2$$

$$\therefore \text{adj}A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

Now,

$$\begin{aligned} A(\text{adj}A) &= \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{Also, } (\text{adj}A)A &= \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \\ &= \begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\text{Hence, } A(\text{adj}A) = (\text{adj}A)A = |A|I.$$

Question 4:

Verify $A(\text{adj}A) = (\text{adj}A)A = |A|I$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

Answer

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$|A| = 1(0-0) + 1(9+2) + 2(0-0) = 11$$

$$\therefore |A|I = 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Now,

$$A_{11} = 0, A_{12} = -(9+2) = -11, A_{13} = 0$$

$$A_{21} = -(-3-0) = 3, A_{22} = 3-2 = 1, A_{23} = -(0+1) = -1$$

$$A_{31} = 2-0 = 2, A_{32} = -(-2-6) = 8, A_{33} = 0+3 = 3$$

$$\therefore \text{adj}A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

Now,

$$\begin{aligned} A(\text{adj}A) &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \end{aligned}$$

Also,

$$\begin{aligned} (\text{adj}A) \cdot A &= \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0+0+0 & 0+2+9 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \end{aligned}$$

Hence, $A(\text{adj}A) = (\text{adj}A)A = |A|I$.

Question 6:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}.$$

we have,

$$|A| = -2 + 15 = 13$$

Now,

$$A_{11} = 2, A_{12} = 3, A_{21} = -5, A_{22} = -1$$

$$\therefore \text{adj}A = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

Question 7:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

We have,

$$|A| = 1(10 - 0) - 2(0 - 0) + 3(0 - 0) = 10$$

Now,

$$A_{11} = 10 - 0 = 10, A_{12} = -(0 - 0) = 0, A_{13} = 0 - 0 = 0$$

$$A_{21} = -(10 - 0) = -10, A_{22} = 5 - 0 = 5, A_{23} = -(0 - 0) = 0$$

$$A_{31} = 8 - 6 = 2, A_{32} = -(4 - 0) = -4, A_{33} = 2 - 0 = 2$$

$$\therefore \text{adj}A = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

Question 8:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}.$$

We have,

$$|A| = 1(-3-0) - 0 + 0 = -3$$

Now,

$$A_{11} = -3 - 0 = -3, A_{12} = -(-3 - 0) = 3, A_{13} = 6 - 15 = -9$$

$$A_{21} = -(0 - 0) = 0, A_{22} = -1 - 0 = -1, A_{23} = -(2 - 0) = -2$$

$$A_{31} = 0 - 0 = 0, A_{32} = -(0 - 0) = 0, A_{33} = 3 - 0 = 3$$

$$\therefore \text{adj}A = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = -\frac{1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

Question 9:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}.$$

We have,

$$\begin{aligned} |A| &= 2(-1-0) - 1(4-0) + 3(8-7) \\ &= 2(-1) - 1(4) + 3(1) \\ &= -2 - 4 + 3 \\ &= -3 \end{aligned}$$

Now,

$$\begin{aligned} A_{11} &= -1 - 0 = -1, A_{12} = -(4 - 0) = -4, A_{13} = 8 - 7 = 1 \\ A_{21} &= -(1 - 6) = 5, A_{22} = 2 + 21 = 23, A_{23} = -(4 + 7) = -11 \\ A_{31} &= 0 + 3 = 3, A_{32} = -(0 - 12) = 12, A_{33} = -2 - 4 = -6 \end{aligned}$$

$$\therefore \text{adj}A = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = -\frac{1}{3} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

Question 10:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}.$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}.$$

By expanding along C_1 , we have:

$$|A| = 1(8-6) - 0 + 3(3-4) = 2 - 3 = -1$$

Now,

$$A_{11} = 8 - 6 = 2, A_{12} = -(0 + 9) = -9, A_{13} = 0 - 6 = -6$$

$$A_{21} = -(-4 + 4) = 0, A_{22} = 4 - 6 = -2, A_{23} = -(-2 + 3) = -1$$

$$A_{31} = 3 - 4 = -1, A_{32} = -(-3 - 0) = 3, A_{33} = 2 - 0 = 2$$

$$\therefore \text{adj}A = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}A = - \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Question 11:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}.$$

We have,

$$|A| = 1(-\cos^2 \alpha - \sin^2 \alpha) = -(\cos^2 \alpha + \sin^2 \alpha) = -1$$

Now,

$$A_{11} = -\cos^2 \alpha - \sin^2 \alpha = -1, A_{12} = 0, A_{13} = 0$$

$$A_{21} = 0, A_{22} = -\cos \alpha, A_{23} = -\sin \alpha$$

$$A_{31} = 0, A_{32} = -\sin \alpha, A_{33} = \cos \alpha$$

$$\therefore \text{adj}A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot \text{adj}A = - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

Question 12:

$$\text{Let } A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}. \text{ Verify that } (AB)^{-1} = B^{-1}A^{-1}$$

Answer

$$\text{Let } A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}.$$

We have,

$$|A| = 15 - 14 = 1$$

Now,

$$A_{11} = 5, A_{12} = -2, A_{21} = -7, A_{22} = 3$$

$$\therefore \text{adj}A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot \text{adj}A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

Now, let $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$.

We have,

$$|B| = 54 - 56 = -2$$

$$\therefore \text{adj}B = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{1}{|B|} \text{adj}B = -\frac{1}{2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix}$$

Now,

$$\begin{aligned} B^{-1}A^{-1} &= \begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix} \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{45}{2} - 8 & \frac{63}{2} + 12 \\ \frac{35}{2} + 6 & -\frac{49}{2} - 9 \end{bmatrix} = \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \quad \dots(1) \end{aligned}$$

Then,

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix} \\ &= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix} \end{aligned}$$

Therefore, we have $|AB| = 67 \times 61 - 87 \times 47 = 4087 - 4089 = -2$.

Also,

$$\begin{aligned} \text{adj}(AB) &= \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \\ \therefore (AB)^{-1} &= \frac{1}{|AB|} \text{adj}(AB) = -\frac{1}{2} \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \quad \dots(2) \end{aligned}$$

From (1) and (2), we have:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Hence, the given result is proved.

Question 13:

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1} .

Answer

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A^2 - 5A + 7I = O$.

$$\therefore A \cdot A - 5A = -7I$$

$$\Rightarrow A \cdot A(A^{-1}) - 5AA^{-1} = -7IA^{-1} \quad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0 \right]$$

$$\Rightarrow A(AA^{-1}) - 5I = -7A^{-1}$$

$$\Rightarrow AI - 5I = -7A^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{7}(A - 5I)$$

$$\Rightarrow A^{-1} = \frac{1}{7}(5I - A)$$

$$= \frac{1}{7} \left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \right) = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

Question 14:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

Answer

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Now,

$$A^2 + aA + bI = O$$

$$\Rightarrow (AA)A^{-1} + aAA^{-1} + bIA^{-1} = O \quad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0 \right]$$

$$\Rightarrow A(AA^{-1}) + aI + b(AA^{-1}) = O$$

$$\Rightarrow AI + aI + bA^{-1} = O$$

$$\Rightarrow A + aI = -bA^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{b}(A + aI)$$

Now,

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

We have:

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{b} \left(\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right) = -\frac{1}{b} \begin{bmatrix} 3+a & 2 \\ 1 & 1+a \end{bmatrix} = \begin{bmatrix} \frac{-3-a}{b} & -\frac{2}{b} \\ -\frac{1}{b} & \frac{-1-a}{b} \end{bmatrix}$$

Comparing the corresponding elements of the two matrices, we have:

$$-\frac{1}{b} = -1 \Rightarrow b = 1$$

$$\frac{-3-a}{b} = 1 \Rightarrow -3-a = 1 \Rightarrow a = -4$$

Hence, -4 and 1 are the required values of a and b respectively.

Question 15:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

For the matrix A show that $A^3 - 6A^2 + 5A + 11I = O$. Hence, find A^{-1} .

Answer

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$\begin{aligned} A^3 &= A^2 \cdot A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} \end{aligned}$$

$$\therefore A^3 - 6A^2 + 5A + 11I$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6 \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$$\text{Thus, } A^3 - 6A^2 + 5A + 11I = O.$$

Now,

$$A^3 - 6A^2 + 5A + 11I = O$$

$$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 5AA^{-1} + 11IA^{-1} = 0 \quad [\text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0]$$

$$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 5(AA^{-1}) = -11(AA^{-1})$$

$$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I) \quad \dots(1)$$

Now,

$$A^2 - 6A + 5I$$

$$\begin{aligned}
 &= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 9 & 2 & 1 \\ -3 & 13 & -14 \\ 7 & -3 & 19 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}
 \end{aligned}$$

From equation (1), we have:

$$A^{-1} = -\frac{1}{11} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

Question 16:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}

Answer

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now,

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{bmatrix} - \begin{bmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I = O$$

Now,

$$A^3 - 6A^2 + 9A - 4I = O$$

$$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 9AA^{-1} - 4IA^{-1} = O$$

[Post-multiplying by A^{-1} as $|A| \neq 0$]

$$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 9(AA^{-1}) = 4(AA^{-1})$$

$$\Rightarrow AAI - 6AI + 9I = 4A^{-1}$$

$$\Rightarrow A^2 - 6A + 9I = 4A^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{4}(A^2 - 6A + 9I) \quad \dots(1)$$

$$A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

From equation (1), we have:

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Question 17:

Let A be a nonsingular square matrix of order 3×3 . Then $|adj A|$ is equal to

- A.** $|A|$ **B.** $|A|^2$ **C.** $|A|^3$ **D.** $3|A|$

Answer **B**

We know that,

$$\begin{aligned} (adj A) A &= |A| I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} \\ \Rightarrow |(adj A) A| &= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} \\ \Rightarrow |adj A| |A| &= |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |A|^3 (I) \\ \therefore |adj A| &= |A|^2 \end{aligned}$$

Hence, the correct answer is B.

Question 18:

If A is an invertible matrix of order 2, then $\det(A^{-1})$ is equal to

- A.** $\det(A)$ **B.** $\frac{1}{\det(A)}$ **C.** 1 **D.** 0

Answer

$$A^{-1} \text{ exists and } A^{-1} = \frac{1}{|A|} adj A.$$

Since A is an invertible matrix,

As matrix A is of order 2, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then, $|A| = ad - bc$ and $\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Now,

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\therefore |A^{-1}| = \begin{vmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{vmatrix} = \frac{1}{|A|^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{1}{|A|^2} (ad - bc) = \frac{1}{|A|^2} \cdot |A| = \frac{1}{|A|}$$

$$\therefore \det(A^{-1}) = \frac{1}{\det(A)}$$

Hence, the correct answer is B.